

# The Visibility Freeze-Tag Problem

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## Abstract

We introduce the *Visibility Freeze-Tag Problem*, a variant of Freeze-Tag in a polygon where line-of-sight, rather than contact, is sufficient to awaken a robot. We show that the problem is NP-hard. In a related problem called the *Line/Point Freeze-Tag Problem*, an awake robot must travel to a pre-specified line (not just any line-of-sight) associated with another robot  $v$  in order to awaken  $v$ . We show that this problem is also NP-hard, and we give a polynomial time  $O(1)$ -approximation algorithm.

## 1 Introduction

In the Freeze-Tag Problem, introduced by Arkin et al. [1], we are given a set of robots at points inside some domain. Initially, all the robots are asleep (“frozen”) except one. That robot can awaken another robot by moving to its position (at uniform speed), and once a robot is awakened, it can move and help to awaken other robots. Robots have full information about the domain and each other’s positions. The goal is awaken all the robots in the shortest time. Basic results from [1] are: the Freeze-Tag problem is NP-hard in graphs (even in a star); there is an  $O(\log n)$ -approximation algorithm in any metric space; and for the case of points in  $\mathbb{R}^d$  there is a PTAS for fixed  $d$ , but the problem is not known to be NP-hard.

In this paper we will prove NP-hardness of the Freeze-Tag Problem in a polygon—in this case, the robots lie at points inside a simple polygon in the plane, and robots can only move inside the polygon. We then consider a variant of this problem called the *Visibility Freeze-Tag Problem* (also dubbed “laser tag” by J. Mitchell) where a robot awakens another robot when it “sees” the other robot inside the polygon, i.e., the line segment between them is inside the polygon. This models freeze-tag with line-of-sight communication. We show that the Visibility Freeze-Tag problem is NP-hard.

Finally, we consider another variant, motivated as follows. In the Visibility Freeze-Tag Problem, if the awake robot  $s$  cannot see robot  $v$ , then there is a line (called a “window”) that goes through  $v$  and some reflex vertex, and  $s$  must move to that line in order to see  $v$ . Ab-

stracting this, we introduce the *Line/Point Freeze-Tag Problem* in the plane where every frozen robot  $v$  has an associated infinite “awakening line”  $l_v$  and  $v$  is awakened when another robot touches its awakening line. Note that we no longer constrain the robots to the interior of a polygon. We prove that the Line/Point Freeze-Tag Problem is NP-hard.

We also give polynomial time approximation algorithms as follows: an  $O(1)$ -approximation for the Line/Point Freeze-Tag Problem, and an  $O(n)$ -approximation for the Visibility Freeze-Tag Problem. Note that the  $O(\log n)$ -approximation algorithm of [1] does not apply to the Visibility Freeze-Tag Problem since awakening times are no longer symmetric: the time for robot  $u$  to awaken  $v$  is not necessarily equal to the time for robot  $v$  to awaken  $u$ .

Complete proofs can be found in [4].

## 2 Main Results

**Theorem 1** *The Freeze-Tag Problem in a polygon is NP-hard.*

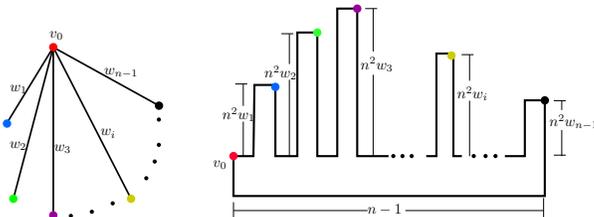


Figure 1: Reduction from the Freeze-Tag Problem in a star (left) to the Freeze-Tag Problem in a polygon (right).

We prove NP-hardness by reducing from the Freeze-Tag Problem for an edge-weighted star graph with the initial awake robot at the center of the star and  $n - 1$  frozen robots, one at each leaf (shown to be NP-hard in [1]). The idea of the reduction is shown in Fig. 1. An edge of the star with weight  $w_i$  becomes a vertical rectangular bay of the polygon of height  $n^2 w_i$ . The  $n^2$  factor ensures that the vertical movement of robots inside the polygon overwhelms the horizontal movement.

**Theorem 2** *The Visibility Freeze-Tag Problem in a polygon is NP-hard.*

We prove this by a slight modification of the previous proof, hiding each robot in a small horizontal bay at the top of its original vertical bay. See Fig. 2.

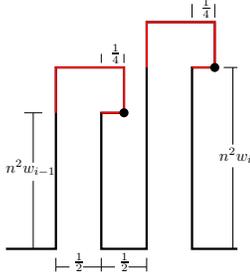


Figure 2: To prove NP-hardness of the Visibility Freeze-Tag Problem, add small horizontal bays (in red).

**Theorem 3** *There is an  $O(n)$ -approximation algorithm for the Visibility Freeze-Tag Problem.*

We use the “shortest inspection path” algorithm of Knauer et al. [3] to find the minimum time  $w(u, v)$  for each robot  $u$  to awaken each robot  $v$  ignoring other robots. This takes  $O(n^2 \log n)$  time. Then we build a shortest path tree for these weights, and ask each robot to visit its children in the tree sequentially.

**Theorem 4** *The Line/Point Freeze-Tag Problem is NP-hard, even for the case where all the awakening lines are horizontal or vertical*<sup>1</sup>.

The reduction is from 3SAT: For each variable  $v_i$  we make a *primary robot* with a horizontal awakening line, and create two sets of *literal robots* with vertical awakening lines — the “true” lines  $T_i$  and the “false” lines  $F_i$  (Fig. 3). By manipulating the position of the awakening lines, we force  $v_i$  to choose either to awaken the lines in  $T_i$  or the lines in  $F_i$ , but not both. For each clause  $C_j$ , we create a *clause robot*  $c_j$  with a horizontal awakening line that can only be awakened by one of three points, which correspond to the literals in the clause. This is the main idea, and the one complication is that we need to add some *relay robots* to help awaken all the robots, not just the ones associated with the true literal robots.

**Theorem 5** *There is an  $O(1)$ -approximation algorithm for the Line/Point Freeze-Tag Problem.*

Following the idea in [2] for searching for a line in the plane, our algorithm asks every robot to follow a logarithmic spiral when it is awakened (Fig. 4). This is a

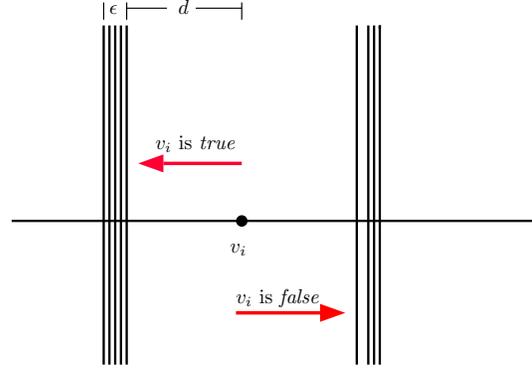


Figure 3: The variable gadget used to prove NP-hardness of the Line/Point Freeze-Tag Problem:  $v_i$  should either awaken the robots to its left or to its right.

somewhat “oblivious” strategy since the robots do not even make use of the information about the locations of other points and lines. Nevertheless, we show that this algorithm achieves a constant approximation factor.

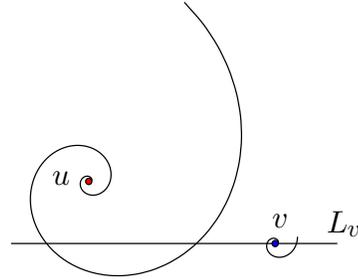


Figure 4: Approximation algorithm for the Point/Line Visibility Freeze-Tag Problem: robot  $u$  follows a logarithmic spiral and when it touches the awakening line  $L_v$  of  $v$  then robot  $v$  begins to follow a logarithmic spiral as well.

## References

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<sup>1</sup>Proved in collaboration with K. Cameron and J. Stacho