Consistent Digital Line Segments in d Dimensions

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Digital geometry plays a fundamental and substantial role in many computer vision applications, for example image segmentation, image processing, facial recognition, fingerprint recognition, and some medical applications. One of the key challenges in digital geometry is to represent Euclidean objects in a digital space so that the digital objects have a similar visual appearance as their Euclidean counterparts. In addition, the digital objects should satisfy some axioms also satisfied by the Euclidean objects. Often it is easy to satisfy the axioms or the visual similarity, but satisfying both simultaneously can be difficult. Representation of Euclidean objects in a digital space has been a focus in research for over 25 years.

In this paper, we consider the problem in the context of representing Euclidean line segments in a digital space. Digital line segments are particularly important to model accurately, as other digital objects depend on them for their own definitions (e.g. convex and star-shaped objects). The digital representation of line segments has been studied under various contexts, but in most representations given in research (including the ones most commonly utilized in computer vision) are such that the intersections of the line segments are not connected, see for example Figure 1. This disconnected intersection may not be desirable for several reasons, for example if the definition of digital line segments has disconnected intersections then the natural definition of a digital convex object may include objects with holes.

In our paper, we consider points in the grid $\mathbb{Z}^d$.

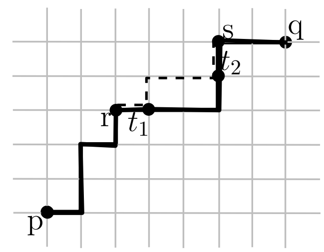

Figure 1: The solid segment from p to q and the dotted segment from r to s have a disconnected intersection.

Let $p = (p^1, p^2, \ldots, p^d)$ and $q = (q^1, q^2, \ldots, q^d)$ be two points in $\mathbb{Z}^d$. We say p and q are neighbors if and only if $\sum_{i=1}^{d} |p^i - q^i| = 1$. For any pair of points $p, q \in \mathbb{Z}^d$ we call the digital line segment from p to q $R_p(q)$. Chun et al.[2] gave the following axioms that arise naturally in Euclidean geometry.

- $(S1)$ *Grid path property:* $R_p(q)$ is the points of a path from p to q in the grid topology.
- $(S2)$ *Symmetry property:* $R_p(q) = R_q(p)$.
- $(S3)$ *Subsegment property:* For every $r, s \in R_p(q)$, we have $R_r(s) \subseteq R_p(q)$.
- $(S4)$ *Prolongation property:* There exists $r \in \mathbb{Z}^d$, such that $r \notin R_p(q)$ and $R_p(q) \subseteq R_p(r)$.
- $(S5)$ *Monotonicity property:* If $p^i = q^i = c$ for any $1 \leq i \leq d$, then every point $r \in R_p(q)$ has $r^i = c$

If a system of digital line segments satisfies the axioms $(S1) - (S5)$, then it is called a *consistent* *digital line segments system* (CDS). The subsegment property (S3) is the main property that separates the study of CDSes from other digital line segment systems. Note the segments in Figure 1 do not satisfy the subsegment property.

Hausdorff Distance. It is easy to give a definition of digital line segments that satisfies all five properties of being a CDS by following the boundary of the axis-parallel bounding box of p and q, but clearly these segments are a poor visual representation of their Euclidean counterparts. The question then becomes one of giving a definition of a CDS such that the *Hausdorff distance* of the digital segments to their Euclidean counterparts as small as possible. Note that the bounding box definition obtains digital line segments with Hausdorff distance $\Omega(n)$ even for $d = 2$, where n is the number of points in the segment. Chun et al.[2] gave a lower bound on the Hausdorff Distance of a CDS of $\Omega(\log n)$ even for $d = 2$. Note that this lower bound is due to property (S3), as it is easy to see that if the requirement of (S3) is removed then digital segments with $O(1)$ Hausdorff distance are easily obtained, for example by using a trivial "rounding" scheme. On the positive side, Christ et al. [1] give a CDS construction with optimal Hausdorff distance of $O(\log n)$ in $\mathbb{Z}^2$. Unfortunately prior to this work nothing is known about constructing CDSes in $\mathbb{Z}^d$ with Hausdorff distance $o(n)$. The proper construction for $\mathbb{Z}^3$ is particularly important as three-dimensional images are commonplace for many extremely important applications in computer vision, such as medical image segmentation.

Our Contribution. Our results take the first significant steps towards obtaining a "good" CDS in $\mathbb{Z}^d$. In our first result, we define a system of consistent line segments in $\mathbb{Z}^d$ for segments coming from any two "slope types", and we show that our construction has Hausdorff distance $O(d^{1.5} \log n)$ which is optimal up to a constant factor for any fixed d. Intuitively, this construction produces consistent segments for a subset of the segments ("half" for $d = 3$), and it is the first construction with Hausdorff distance $o(n)$ even for segments coming from one slope type. In our second result, we give the first non-trivial CDS in $\mathbb{Z}^d$ (i.e. for *all* line segments). Unfortunately there are some segments with Hausdorff distance $\Omega(n)$, but our construction provides insight into how CDSes in d dimensions can be constructed.

High Level Idea. The *slope type* of $p, q \in \mathbb{R}^d$ is the sign vector $(\sigma_1, \sigma_2, \ldots, \sigma_d) \in \{+1, -1\}^d$ where $\sigma_i = +1$ if $p^i \leq q^i$ and is -1 otherwise. The slope type of $R_p(q)$ is defined to be the same as that of the slope type of p and q. Intuitively, line segments of the same slope type "move in the same direction".

Suppose we have $d = 3$, and consider segments $R_p(q)$ and $R_{p'}(q')$ of different slope types, and suppose $R_p(q)$ and $R_{p'}(q')$ have a disconnected intersection. Let t_1 be the last point in both segments before separating, and let t_2 be the first point at which they meet again (similar to Figure 1). The key observation is to notice that t_1 and t_2 must lie in the same two-dimensional "plane". We ensure the the projection of all our three-dimensional segments onto this two-dimensional plane is exactly the CDS given by Christ et al. [1]. It follows that our segments move from t_1 to t_2 according to a segment in a two-dimensional CDS, and this ensures that our three-dimensional segments are consistent.

References

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[2] J. Chun, M. Korman, M. Nöllenburg, and T. Tokuyama. Consistent digital rays. *Discrete & Computational Geometry*, 42(3):359–378, 2009.