Faster Reductions from Straight Skeletons to Motorcycle Graphs*

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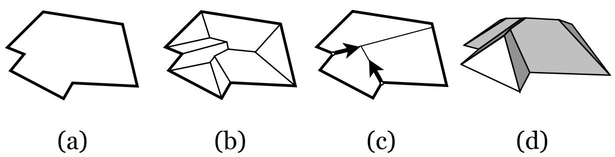

Figure 1: (a) A polygon, (b) its straight skeleton, (c) its induced motorcycle graph, and (d) its straight skeleton roof.

1 Introduction

The straight skeleton of a simple polygon (Fig. 1b) is a tree-like structure that subdivides its interior into regions. It was first defined for polygons [1] and was later generalized to planar straight line graphs (PSLGs) [2].

The fastest algorithms for the straight skeleton construct a lifting of it into 3D called the *straight skeleton roof* (Fig. 1d). This was shown by Eppstein and Erickson [7] to be the lower envelope of a particular set of partially infinite 3D strips called *slabs*. Each edge of the polygon (or PSLG) has one slab on which its roof face is defined; however the slabs are defined in reference to the straight skeleton and there is no known way to compute them independently of computing the entire straight skeleton. They also modeled the main computational hurdle by a structure called the *motorcycle graph*. The basic idea is to place *motorcycles* in the plane and let them move along linear trajectories while laying down a track behind them. If a motorcycle encounters the track of another it crashes. Run this process out to infinity and the result is the motorcycle graph. Cheng and Vigneron [5] defined an *induced motorcycle graph* for a polygon (Fig. 1c) in which motorcycles are placed at every reflex vertex of the polygon (with certain velocities), and the edges of the polygon are treated as additional obstacles. From this they defined a slightly different slab set that depends only on the induced motorcycle graph for its definition. Once the induced motorcycle graph is known, computing the straight skeleton reduces to computing a lower envelope of slabs.

Currently, the fastest motorcycle graph algorithms are $O(n^{4/3+\epsilon})$ time in the non-degenerate case [10] and $O(n^{17/11+\epsilon})$ in the degenerate case [7]. The fastest reductions from the straight skeleton to the motorcycle graph are the recent $O(n(\log n)\log r)$ time algorithm for a polygon (possibly with holes) with r reflex vertices [4] and $O(n^2 \log n)$ time algorithm for PSLGs [8]. In the case of polygons (with holes) the fastest way to compute the straight skeleton is to first compute its motorcycle graph and then apply a reduction. For PSLGs, however, the fastest algorithm is $O(n^{17/11+\epsilon})$ [7] and does not use the motorcycle graph. We show:

Theorem 1. *Given its induced motorcycle graph, the straight skeleton can be computed in $O(n \log n)$ time for an n-vertex polygon and $O(n(\log n)\log m)$ for an n-vertex PSLG with m connected components.*

Used in conjunction with the fastest motorcycle graph algorithm this speeds up the computation of the straight skeleton for polygons and PSLGs.

2 Overview

Our algorithm computes the straight skeleton roof for a polygon (without holes) using a divide

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and conquer approach similar to Shamos and Hoey's famous algorithm for Voronoi diagrams [9]. A similar approach was recently employed in [3] for the straight skeleton of monotone polygons. However in both cases, the two resulting surfaces are terrains, and the intersection between the two is a simple path, which is not true in our case. We would like to do the following: subdivide the polygon into two, compute the lower envelope of the slabs for each subchain, and merge the result. The problem with this is twofold, first the complexity of the lower envelope of the slabs for a single chain may be $\Omega(n^2\alpha(n))$[1] [6]; and second, the intersection between the two lower envelopes may not be a simple path.

We define a surface called a *partial roof* that captures enough of the lower envelope at intermediate steps that we can compute the straight skeleton roof at the end, but does not try to maintain the entire lower envelope. The main properties satisfied by any partial roof for a particular sub-chain C are, *face containment*–each face of the partial roof geometrically contains the face of the final roof supported by its slab; and *edge containment*–if the final straight skeleton roof contains an edge e on the intersection of slabs s_1 and s_2, and the base edges for s_1 and s_2 are in C, then the partial roof contains an edge e' on the intersection of s_1 and s_2 that geometrically contains e.

From these we show that the only structure satisfying the definition of a partial roof for an entire polygon is the straight skeleton roof. We then give an algorithm for merging partial roofs for sub-chains that are incident at a vertex. We then adapt the subdivision procedure from [4] to extend our result to PSLGs.

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[1] Where $\alpha(n)$ denotes the inverse Ackermann function.