GEOMETRIC RIGIDITY OF GRAPHS ON THE TORUS

ELISSA ROSS, BRIGITTE SERVATIUS AND HERMAN SERVATIUS

A *gain graph* is a graph G whose edges are labeled invertibly by the elements of a group Γ [2]. We denote a gain graph by $\langle G, g\rangle$, where $g : E(G)^+ \to \Gamma$ is the labelling of the forward-directed edges. If the directed edge e has label g_e, then the other direction of the edge has label g_e^{-1} (see Figure 1). When $\Gamma = \mathbb{Z}^2$, the gain graph $\langle G, g\rangle$, together with a *realization* p of the vertices $V(G)$ in the plane, provides a description of an infinite periodic graph realized in $\mathbb{R}^2$ (see Figure 2). We view p as a realization of the vertices on the *fixed torus*: the topological torus formed by identifying the opposite sides of the square $[0, 1] \times [0, 1]$.

The present work is concerned with three objects:

(1) An infinite periodic graph, realized in $\mathbb{R}^2$ (*infinite periodic framework*). Denote by $(\langle G, g\rangle, p)$.
(2) Its underlying *gain graph:* $\langle G, g\rangle$, where G is the quotient graph under translational symmetry, and g is a labelling of the edges by elements of $\mathbb{Z}^2$.
(3) The *geometric gain framework*: The quotient graph G realized in $\mathbb{R}^2$ with same vertex geometry as in the infinite periodic graph, but now with straight edges, and no gains. Denote by (G, p).

An *infinitesimal motion* of an infinite periodic framework is an assignment $u : V(G) \to \mathbb{R}^2$ of infinitesimal velocities to the vertices of the gain graph $\langle G, g\rangle$ on the fixed torus such that the edges lengths of the framework are (infinitesimally) preserved. If an infinitesimal motion preserves the distances between *all* pairs of vertices, it is called *trivial*. Such an infinitesimal motion is forced to be *periodic*, since it is defined on orbits of vertices and edges in the quotient graph.

Theorem 1. *Fix one vertex of the infinite periodic framework, and fix the corresponding vertex of the geometric gain graph. If there is a nontrivial infinitesimal motion of the infinite periodic framework $(\langle G, g\rangle, p)$, then we can "shrink" the gains to obtain an infinitesimal motion of the geometric gain framework (G, p). The infinitesimal motion of the geometric gain framework is either a rotation about the fixed vertex, or is non-trivial.*

In other words, every infinitesimal motion of the infinite periodic framework corresponds to an infinitesimal motion of the geometric gain framework. The converse is not true. However, it is an interesting question to determine which motions of the geometric gain graph lift to motions of the infinite periodic framework.

The following example of a generically rigid graph which is geometrically flexible provides an illustration of our methods. By [1] K_4 is generically rigid on the fixed torus. In the geometric position with coordinates given below, it is flexible. We show that the infinitesimal motion on the periodic framework can be shrunken to an infinitesimal motion of the geometric gain graph.

Date: October 19, 2014.

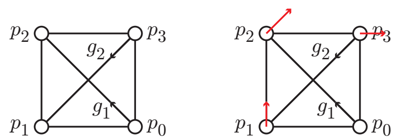

FIGURE 1. Gain graph $\langle G, g\rangle$. Unlabeled edges have gain $(0,0)$. Red arrows indicate an infinitesimal rotation lifting to $(\langle G, g\rangle, p)$

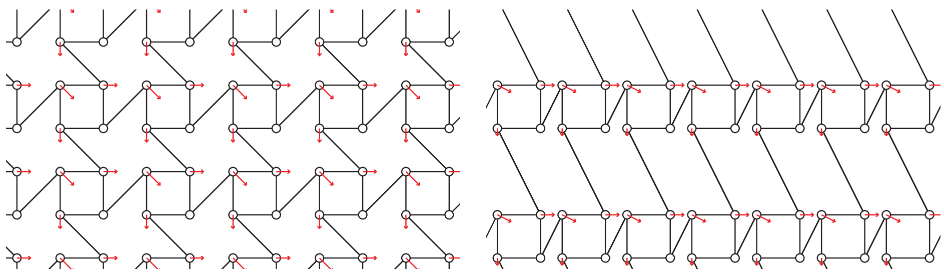

FIGURE 2. The periodic framework corresponding to the gain graph of Figure 1 (left). The "shrink" of this infinitesimal motion (red arrows) in progress (right).

Let the vertices be placed as follows:

$$p_0 = (0.5, 0),\ p_1 = (0,0),\ p_2 = (0, 0.5),\ p_3 = (0.5, 0.5).$$

Let $g_1 = (1,0)$, $g_2 = (-1,0)$. The initial motion assignment u is:

$$u = (\begin{array}{cccccccc} 0 & 0 & 0 & -a & a & -a & a & 0 \end{array}).$$

We shrink this motion as follows: for $t \in [-1,1]$, let g_1 and g_2 be written in homogeneous coordinates as $g_1(t) = [\frac{1-t}{2} : 0 : 1]$ and $g_2(t) = [0 : \frac{1-t}{2} : t]$. Note that as t passes through zero, the y coordinate of $g_2(t)$ passes through infinity. If we demand that $u_0(1) = u_0(-1)$, and $u_3(1) = u_3(-1)$ (i.e. that the initial velocities at the vertices v_0 and v_3 remain fixed), then as we shrink the gains, the infinitesimal motions at the vertices v_1 and v_2 are as follows:

$$u(t) = (\begin{array}{cccccccc} 0 & 0 & 0 & at & a & at & a & 0 \end{array}).$$

In particular, when $t = 1$ and $g_1(1) = g_2(1) = (0,0)$, then

$$u(1) = (\begin{array}{cccccccc} 0 & 0 & 0 & a & a & a & a & 0 \end{array}),$$

which is an infinitesimal rotation of $(\langle G, g\rangle, p)$ about the fixed vertex p_0.

REFERENCES

[1] Elissa Ross, *Geometric and combinatorial rigidity of periodic frameworks as graphs on the torus.* Ph.D. Thesis York University (Canada). ProQuest LLC, Ann Arbor, MI, 2011. 336 pp. ISBN: 978-0494-75689-8

[2] T. Zaslavsky. Biased graphs. I. Bias, balance, and gains. *J. Combin. Theory Ser. B*, 47:32 – 52, 1989.

MATHEMATICAL SCIENCES, WORCESTER POLYTECHNIC INSTITUTE, WORCESTER MA 01609-2280
E-mail address: bservat@wpi.edu