FPTAS for Minimizing Earth Mover's Distance under Rigid Transformations and Related Problems*

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1 Introduction

In this paper, we study the problem (denoted as EMDRT) of minimizing the earth mover's distance between two sets of weighted points A and B (with size n and m, respectively) in a fixed dimensional $\mathbb{R}^d$ space under rigid transformation. In EMDRT, each point in A and B is associated with a nonnegative weight, and the objective is to determine the best rigid transformation $\mathcal{T}$ for B so that the earth mover's distance (EMD) between A and $\mathcal{T}(B)$ is minimized, where EMD measures the minimum transportation cost between the two point sets. EMDRT is an important problem in both theory and applications. In theory, it is a natural generalization of the bipartite matching problem (*i.e.,* from one-to-one matching to many-to-many matching) and is a powerful model for a number of other matching or partial matching problems. For instance, if all points in A and B have unit weight, EMDRT becomes a one-to-one matching problem (*e.g.,* the congruent and alignment problem). If all points in A have unit weight and all points in B have infinity weight, EMDRT becomes a many-to-one matching problem (*i.e.,* the Hausdorff distance matching problem). In applications, EMDRT has connections to many EMD (or its variants such as *Proportional Transportation Distance (PTD)*) based problems in pattern recognition and computer vision [2, 9], and can be used to solve the challenging alignment problem for rigid objects and detect similarity between multi-dimensional point sets.

A number of results exist for EMDRT and its related problems. Cabello *et al.* [4] presented several approximation results in $\mathbb{R}^2$ space; particularly, they gave a $(2+\epsilon)$-approximation solution for the 2D EMDRT problem, and a $(1+\epsilon)$-approximation solution for a special case in which only translation is allowed. Later, Klein and Veltkamp [8] introduced a few improved results by using *reference point*, and achieved an $O(2^{d-1})$-approximation for EMDRT in $\mathbb{R}^d$ space. It has been an open problem for quite some time to achieve PTAS for this problem [4]. For static points (*i.e.,* without transformation), several approximation algorithms for computing EMD were presented in [10]. For the related *alignment* (also called *geometric matching*) problem, there is a long and rich history, and some early results on this problem can be found in the survey paper by Alt and Guibas [1].

Our results. In this paper, we present the first FPTAS algorithm for EMDRT in any fixed dimensional space. Our result is based on a few new techniques, such as *Sequential Orthogonal Decomposition (SOD)* and *Optimum-Guided-Base (OGB)*. SOD decomposes a rigid transformation into a sequence of primitive operations which enables us to accurately analyze how the transportation flow changes in a step-by-step fashion during the whole process of rigid transformation and therefore have a better estimation on the quality of solution. OGB enables us to use some information of the unknown optimal solution to select some critical points which partially define the rigid transformation. We show that although OGB cannot be explicitly implemented, its result can actually be implicitly obtained. A major advantage of OGB is that it can help us to significantly reduce the search space. Consequently, our FPTAS runs in roughly $O((nm)^{d+2}(\log nm)^{2d})$ time, which is close to (*i.e.,* matches the order of magnitude of the degree of) the lower bound $\Omega(\max\{m,n\}^{O(d)})$ on the running time of any PTAS for EMDRT [3].

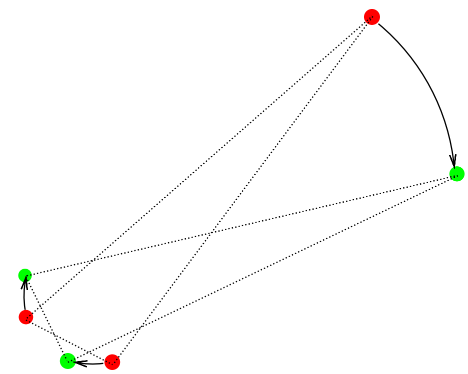
Our techniques for EMDRT can be extended to several related problems, such as the problem of minimizing EMD under similarity transformation (i.e., rigid transformation plus scaling) and the alignment problem, and achieve an FPTAS for each of them. For the alignment problem, we consider three different cost metrics, the l_∞ sense metric and l_1 sense metric in the one-to-one matching case, and the Hausdorff distance metric in the one-to-many matching case. Our result for the Hausdorff distance metric is an FPTAS, while existing results [5, 7] are only pseudo-PTAS (*i.e.,* depending on the spread ratio).

* A preliminary version of this paper has appeared in the 21st European Symposium on Algorithms (ESA 2013) [6].

2 Overview of Our Approach

The main task for achieving an FPTAS is to find a good approximation of the optimal rigid transformation $\mathcal{T}_{opt}$ for B, which is equivalent to identify d points R (called *reference system*) from A and another d sorted points (called *base*) from B and determine a rigid transformation $\mathcal{T}$ to map points in the base to the neighborhoods of R.

Our approach consists of two main steps: (1) Design a polynomial-approximation algorithm to compute an upper bound of the optimal objective value, and (2) use the upper bound to derive an FPTAS. In both steps, directly searching for $\mathcal{T}$ in the rigid transformation space could be very costly. Our idea is to introduce a new technique called *Sequential Orthogonal Decomposition (SOD)*, which decomposes the rigid transformation into a sequence of d primitive operations (*i.e.,* a translation and $d-1$ one-dimensional rotations). One important property of SOD is that its outcome is independent of the initial position of B and depends only on the choice of R and the base. This enables us to assume that B is initially located at $\mathcal{T}_{opt}(B)$. Another important property of SOD is that it allows us to analyze, in a step by step fashion, how the transportation flow (*e.g.,* the bottleneck flow) changes when B moves from $\mathcal{T}_{opt}(B)$ to $SOD(B)$. This gives us an accurate estimation on the quality of solution.



(a)

Fig. 1: A small rotation of a non-dispersed base (*i.e.,* the two red points at the left bottom) causes a large error for a faraway point at the right top.

The quality of the rigid transformation $\mathcal{T}$ determined by SOD depends on the reference system R and the base. To find a good R, we first build a grid in the neighborhood of each point in A and then consider as R all possible subsets (with cardinality d) of A and its grid points. To find a good base, a key problem is that a small error in the rotation could cause some point in B to move a long distance and therefore introduce a large error (see Fig. 1). To avoid this problem, we select the base as a set of d ordered points in B which are as "dispersed" as possible (since other choices cause larger error). We consider two types of dispersions. Type 1 dispersion is based on the weighted distance (with weight β_j) between each point q_j and the flat spanned by all points appeared before q_j in the sorted order of the base. Type 2 dispersion considers not only the weighted distance, but also the distance between each q_j and all points in A in the optimal solution. We show that although such distances cannot be explicitly computed (as it depends on the optimal solution), it can actually be implicitly obtained. A major advantage of the second type of dispersion is that it enables us to significantly improve the running time.

References

1. H. Alt, and L. Guibas, *Discrete geometric shapes: matching, interpolation, and approximation*, J.-R. Sack, J. Urrutia (Eds.), Handbook of Computational Geometry, Elsevier, Amsterdam, pp. 121–153, 1999.
2. S. Cohen, *Finding color and shape patterns in images*, PhD thesis, Stanford University, Department of Compute Science, 1999.
3. S. Cabello, P. Giannopoulos and C. Knauer, *On the parameterized complexity of d-dimensional point set pattern matching*, Inf. Process. Lett. 105(2): 73–77, 2008.
4. S. Cabello, P. Giannopoulos, C. Knauer, and G. Rote, *Matching point sets with respect to the earth mover's distance*, Computational Geometry: Theory and Applications, Vol. 39, No. 2, pp. 118–133, 2008.
5. D. E. Cardoze and L. J. Schulman, *Pattern matching for spatial point sets*, Proc. 39th Annual IEEE Symposium on Foundations of Computer Science (FOCS'98), pp. 156–165, 1998.
6. H. Ding and J. Xu, *FPTAS for minimizing earth mover's distance under rigid transformations*, Proc. 21st Annual European Symposium (ESA'13), pp. 397-408, 2013.
7. M. Gavrilov, P. Indyk, R. Motwani and S. Venkatasubramanian, *Combinatorial and experimental methods for approximate point pattern matching*, Algorithmica, Vol. 38, No. 1, pp. 59–90, 2004.
8. O. Klein and R. C. Veltkamp, *Approximation algorithms for computing the earth mover's distance under transformations*, Proc. 16th International Symposium on Algorithms and Computation (ISAAC'05), pp. 1019–1028, 2005.
9. Y. Rubner, C. Tomasi and L. J. Guibas, *The earth movers distance as a metric for image retrieval*, Int. J. of Comp. Vision, 40(2), pages 99–121, 2000.
10. R. Sharathkumar and P. K. Agarwal, *Algorithms for the transportation problem in geometric settings.* SODA 2012, 306-317.