

A Subdivision Approach to Weighted Voronoi Diagrams

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1 Introduction

Voronoi diagrams are a central topic in Computational Geometry. They can be generalized in many ways. Suppose S is a finite set of geometric objects and there is a notion of Voronoi diagram $Vor(S)$ that we want to compute. One approach to designing algorithms for $Vor(S)$ is to (1) assume some general properties of S , and (2) describe an abstract algorithm based on the Real RAM model that has some postulated capability. In \mathbb{R}^2 , (1) might say that the bisector of two objects is a simple infinite curve, and (2) might assume the ability to compute the intersection of any pair of bisectors. For example, see [2] in \mathbb{R}^2 and [1] for semi-algebraic convex objects in \mathbb{R}^3 . On the other hand, if we want to implement these algorithms, these abstract algorithms often pose immense barriers. An example of this phenomenon is the fact that there is currently no implementable algorithm for the Euclidean Voronoi diagram of a set S of polyhedral objects. See [8] for a discussion of the issues.

This is a strong motivation for exploring models of computation other than the Real RAM. This paper continues our exploration of approximate numerical models: we use a specific form of this model, based on the subdivision paradigm [7]. The first subdivision algorithm for the Voronoi diagram of polyhedral solids is from [4]. Other subdivision algorithms for Voronoi diagrams are surveyed in [8].

A key observation about such models is that they are relatively easy to implement and very flexible. That is in marked contrast with exact algorithms which often require completely different algorithms when the setting is slightly generalized. This phenomenon is demonstrated in the present paper. In [8], we described and implemented an algorithm for the Euclidean Voronoi diagram for a set of pairwise disjoint set of polygons, assuming suitable non-degeneracy conditions. In this paper, we show how to extend such an algorithm to handle weighted polygons. This generalizes previously studied classes of weighted Voronoi diagrams: multiplicative or additive weights [5] and anisotropic weights [3]. Such Voronoi diagrams have quartic curves. But if we drop the additive weights, the resulting Voronoi diagram has only conic curves, and moreover all three types of conics (parabola, ellipse and hyperbola) do appear. These Voronoi diagrams appear to be new, and offer a flexible modeling tool in applications.

1.1 Overview

Our subdivision approach to Voronoi diagrams follows [8]. The basic idea is to design **soft predicates** [7] and to use these to construct the correct combinatorial structure of the Voronoi diagrams. Under suitable non-degeneracy conditions, the algorithm terminates. As in [8], the issue of degeneracy is left for future treatment. In this paper, we view the subdivision method as having 2 phases: (I) Subdivide the initial region-of-interest (a box $B_0 \subseteq \mathbb{R}^2$) until some predicate holds. (II) Construct the desired Voronoi diagram. Our goal is to compute a cell complex that is ε -isotopic to $Vor(S)$ (see [8]). If $\varepsilon = \infty$ (the default), then we only care about topological correctness but not about geometric accuracy. By setting $\varepsilon > 0$ small enough, we can compute the diagram to any desired accuracy. Our C++ implementation and datasets are distributed with the **Core Library**¹.

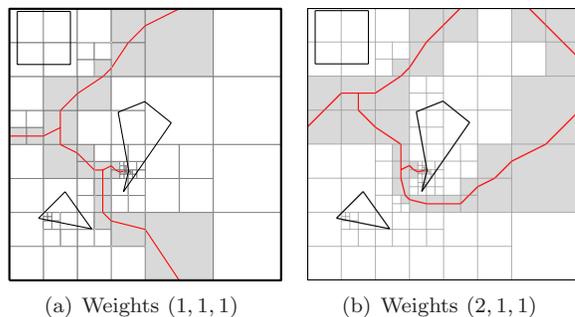


Figure 1: Voronoi diagram of three polygons: un-weighted and weighted

Figures 1–2 show the weighted Voronoi diagram we compute for a set S with three polygons (triangle, square, pentagon) with various multiplicative weights. If their weights are (1, 1, 1) then this corresponds to their usual Euclidean Voronoi diagram, as shown in Figure 1(a). By choosing a small ε , we obtain a highly accurate representation as in Figure 2(b).

A **polygonal set** is a closed set $P \subseteq \mathbb{R}^2$ whose boundary ∂P can be partitioned into a finite set of (boundary) **features** which are either **corners** (points) or **edges** (open line segments). A (geometric) **object** is any simply-connected polygonal set P , with an associated

¹ <http://cs.nyu.edu/exact/core/download/core/>.

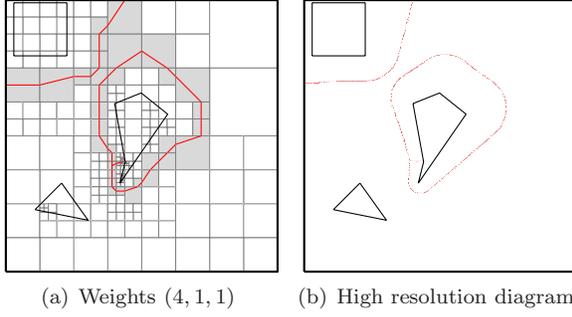


Figure 2: Voronoi diagram of three polygons with Weights (4, 1, 1)

multiplicative weight $\mu_P > 0$, an additive weight $\alpha_P \geq 0$, and a metric tensor M_P which is a 2×2 symmetric positive definite matrix. We say P is **unweighted** if $\mu_P = 1$, $\alpha_P = 0$ and $M_P = I$ (the identity matrix). For any point p and metric tensor M , define the M -norm to be $\|p\|_M := \sqrt{p^T M p}$ where p^T is the transpose of p . The Euclidean norm is $\|p\|_I$, simply denoted $\|p\|$.

Let $\Omega(P)$ denote the set of boundary features of P . Each $f \in \Omega(P)$ has associated μ_f, α_f, M_f which are inherited from P . If X is a feature or an object, and p a point, let the **X -projection** of p be the point p_X in \overline{X} (the closure of X) such that $\|p - p_X\|_{M_X}$ is minimum. Note that if X is a corner, then $p_X = X$. Define the $d_f : \mathbb{R}^2 \rightarrow \mathbb{R}$ as $d_f(p) := \frac{\|p - p_f\|}{\mu_f} - \alpha_f$. The **P -distance function** $d_P : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $d_P(q) = -\alpha_P$ if $q \in P$; otherwise $d_P(q) = \min \{d_f(q) : f \in \Omega(P)\}$. The **extent** of P is $E(P) := \{q \in \mathbb{R}^2 : d_P(q) \leq 0\}$.

Henceforth, fix a set S of objects with pairwise disjoint extents. In this abstract, we assume the isotropic case (so $M_P = I$) and let $K = \max \{1/\mu_P : P \in S\}$. Let $\text{dom}(S) := \{q \in \mathbb{R}^2 : (\forall P \in S) d_P(q) > 0\}$. The **clearance** of $q \in \text{dom}(S)$ is $\text{Clr}(q) := \min \{d_P(q) : P \in S\}$. The (set-theoretic) **Voronoi diagram** $\overline{\text{Vor}}(S)$ comprises those $q \in \text{dom}(S)$ such that there exist two $f, g \in \Omega(S)$ with $d_f(q) = d_g(q) = \text{Clr}(q)$. Given a box $B_0 \in \mathbb{R}^2$ we wish to compute a cell-complex whose support is isotopic to $\overline{\text{Vor}}(S) \cap B_0$ ([8]).

We now describe our soft predicate: Let $B \subseteq \mathbb{R}^0$ be a box with center m_B and circumradius r_B . The **feature set** $\phi(B)$ of B comprises $f \in \Omega(S)$ such that for some $p \in B$, $d_f(p) = \text{Clr}(p)$. We also define $\tilde{\phi}(B) := \{f \in \Omega(S) : d_f(m_B) \leq \text{Clr}(m_B) + 2K \cdot r_B\}$. It can be shown that $\tilde{\phi}(B)$ contains $\phi(B)$. When we split B into four children, we see that $\tilde{\phi}(B') \subseteq \tilde{\phi}(B)$ for each child B' of B .

The soft predicate in [8] includes a “special” test developed for parabolic Voronoi curves. In this paper, we must use the more general “Plantinga-Vegter predicate” [6] that is applicable to any curve defined by $F(x, y) = 0$ where $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^1 -function. Assume that

$\square F : \square \mathbb{R}^2 \rightarrow \square \mathbb{R}$ is an interval function corresponding to F , and similarly for $\square F_x$ and $\square F_y$ where F_x, F_y are partial derivatives with respect to x and y . This predicate, defined as $C_{pv}(B)$ is true iff either $0 \notin \square F(B)$ or $0 \notin (\square F_x(B))^2 + (\square F_y(B))^2$. In our case, $F = 0$ is the bisector curve defined by two features f, g ; thus, the function is $F(x, y) = F^{f,g}(x, y) := d_f(q) - d_g(q)$ where $q = (x, y)$. The predicate $C(B)$ is true when $\tilde{\phi}(B)$ has at most 3 features, and for each $f, g \in \tilde{\phi}(B)$, we want the Plantinga-Vegter $C_{pv}(B)$ to hold at B .

Finally, in the construction phase, we must be able to confirm that a given B with 3 features actually contains a Voronoi vertex. There is a general solution for this test, but here we exploit the special nature of our diagrams. We plan to extend our implementation to incorporate anisotropic diagrams. It should be possible to extend these methods to 3D and beyond.

References

- [1] P. Harrington, C. O. Dúnlaing, and C. K. Yap. Optimal Voronoi diagram construction for convex sites in three dimensions. *IJCGA*, 17(6):555–593, 2007.
- [2] R. Klein. *Concrete and abstract Voronoi diagrams*. Lecture Notes in Computer Science, No. 400. Springer-Verlag, Berlin, 1989.
- [3] F. Labelle and J. R. Shewchuk. Anisotropic voronoi diagrams and guaranteed-quality anisotropic mesh generation. In *19th SoCG*, pp. 191–200, New York, 2003. ACM.
- [4] D. Lavender, A. Bowyer, J. Davenport, A. Wallis, and J. Woodwark. Voronoi diagrams of set-theoretic solid models. *IEEE Computer Graphics and Applications*, 12(5):69–77, 1992.
- [5] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*. John Wiley & Sons, 2000.
- [6] S. Plantinga and G. Vegter. Isotopic approximation of implicit curves and surfaces. In *Proc. Symp. on Geometry Processing*, pp. 245–254, New York, 2004.
- [7] C. Wang, Y.-J. Chiang, and C. Yap. On Soft Predicates in Subdivision Motion Planning. In *29th SoCG*, pp. 349–358, 2013.
- [8] C. Yap, V. Sharma, and J.-M. Lien. Towards Exact Numerical Voronoi diagrams. In *9th ISVD*, pp. 2–16. IEEE, 2012.