Applications of Persistent Homology to Simplicial Ricci Flow

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1 Abstract

We apply the methods of persistent homology to investigate singularity formation in a selection of two and three–dimensional geometries evolved by simplicial Ricci flow, an unstructured mesh formulation of Hamilton's Ricci flow. To implement persistent homology, we construct a triangular mesh for a sample of points. The scalar curvature along the edges of the triangulation, computed as an average of scalar curvatures at the endpoints of the edges, serves as a filtration parameter at each time step. We present and analyze the results of the application of persistent homology to a two–dimensional rotational solid that collapses and three–dimensional dumbbells that manifest neckpinch singularities. We compare the appearance of critical geometric phenomena in these models with the results of the application of persistent homology and conclude via various resolutions that persistent homology does indicate geometric criticality. Finally, we discuss the interpretation and implication of these results and future applications.

2 Overview

Persistent homology (PH) [1] is a method based in algebraic topology that is useful for analyzing data generated by nonlinear processes. The approach involves building a one–parameter family of triangulable topological spaces, called a *filtration*, about the data points and tracking the appearance and disappearance of connected components, tunnels, cavities and their higher–dimensional analogues as the parameter is increased. The parameter, called a *filtration parameter*, is a function on various elements of an appropriate simplicial complex (e.g., scalar curvature measured at points of a smooth manifold put into a simplicial complex). *Persistence intervals* measure the extent to which these topological features persist over a range of values of the parameter. Such values are called birth (the first value at which a feature appears) and death (the final value at which a feature appears).

Ricci flow (RF) has been used in the classification of two and three dimensional geometries [2]. Let M be a Riemannian manifold with metric $\mathbf{g}$. Then, the *Ricci flow equation* is

$$g^{ac}\, \dot{g}_{cb} = -2\, Rc^a_b,$$

where Rc^a_b are the components of the Ricci tensor. The Ricci tensor measures the deviation of the volume of a geodesic ball in M from that of a corresponding ball in Euclidean space. The RF equation diffuses irregularities in the metric on a Riemannian manifold and is useful for finding constant curvature metrics.

Simplicial Ricci flow (SRF) [3] seeks to provide a discrete analogue of this process for piecewise–flat (PF) manifolds. Piecewise–flat manifolds are of interest due to their appearance in problems such as complex networks and imaging. The simplicial form of the RF equations depends on the edge lengths of a d-dimensional PF manifold, $\mathcal{K}$, and the inherited structure of its dual circumcentric lattice, $\mathcal{K}^*$. Each vertex of $\mathcal{K}^*$ is located at the circumcenter of its corresponding d-simplex in $\mathcal{K}$. Adjacent circumcentric vertices are connected with a dual edge $\lambda \in \mathcal{K}^*$. The dual edge λ is associated with a $(d-1)$-dimensional simplex in $\mathcal{K}$. We define the *dual–edge simplicial Ricci flow equation* for any compact, PF simplicial geometry, $\mathcal{K}$, as an equation for each edge, λ, in the circumcentric dual lattice, $\mathcal{K}^*$,

$$\frac{1}{\lambda}\frac{\partial \lambda}{\partial t} = -Rc_\lambda,$$

where Rc_λ is the Ricci tensor associated to the circumcentric dual edge λ.

Using the results of [4], these dual–edge SRF equations may be expressed, for any compact, PF simplicial geometry, $\mathcal{K}$, as an equation for each edge, $\ell \in \mathcal{K}$,

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$$\left\langle \frac{1}{\lambda}\frac{\partial \lambda}{\partial t}\right\rangle_{\ell} = -Rc_{\ell},$$

where $Rc_{\ell} = \langle Rc_{\lambda}\rangle$ is the Ricci tensor associated to the simplicial edge ℓ, and the brackets denote a certain volume-weighted average [3]. We refer to these equations as the *SRF equations*.

The goal of this work is to use PH to detect the formation of singularities in SRF by identifying trends and tracking topological features in the data. We consider metrics on S^2 and S^3 as shown in Figure 1.

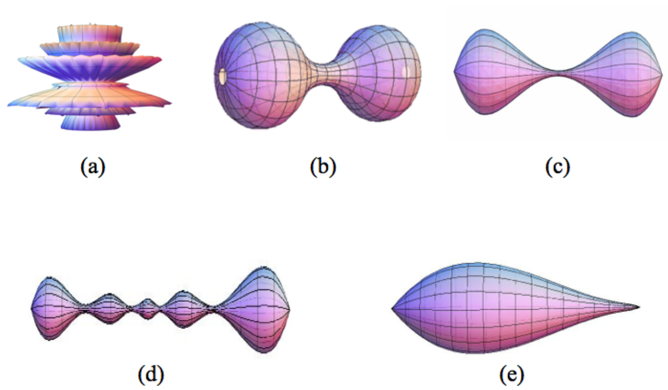

Figure 1: (a) Dimpled 2–sphere; (b) Symmetric Dumbbell; (c) Another Symmetric Dumbbell; (d) Dimpled Dumbbell; (e) Degenerate Dumbbell

The singularities that form include collapse to a round point in S^2 and neck pinch in S^3. Of the neck pinch singularities, there are two types: rapidly forming and slowly forming [5]. Because previous experience has indicated stiffness in some of the SRF algorithms, we exploit the results of [6] which show that the discrete and continuum representations of RF are shown to be equivalent so that one can either begin with a PF manifold $\mathcal{K}$, then take the continuum limit or begin with a continuum manifold, then discretize. As it allows for use of a faster algorithm, we numerically evaluate (hence discretize) the continuum RF equations in a **Mathematica** algorithm for the considered examples on S^2 and S^3, then build a suitable triangulation of the data for analysis via **Perseus**. We filter on the scalar curvature of edges for a discrete sample of times.

3 Summary of Results

We will discuss the results that will appear in [7]. At each time step, **Perseus** prints out tables of persistence intervals for each dimension and a table of Betti numbers that allows us to view features being born and that die off. The symmetry of our models means that we have only dimensions 0 and 1. We find for the different models three classifications of lengths of persistence intervals: negligible finite, finite, and infinite. The tables for connected components display all of these types of intervals; those for the tunnels only have finite intervals. The first classification consists of extremely short intervals representing features that do not persist over many values of scalar curvature. The second classification consists of longer finite intervals, one or more orders of magnitude longer than the first class of finite intervals. The third classification is the presence of infinite persistence intervals associated to low values of scalar curvature.

The dimpled 2–sphere uniformizes in curvature before collapsing to a point. The persistence intervals reflect this transition as most of the finite intervals vanish at the time of uniformization, returning only near the end of the evolution. Infinite intervals with increasing birth scalar curvature at each time step appear in the tables of persistence intervals for dimension 0. This corresponds with the formation of a singularity.

For the dimension 0 persistence tables, the symmetric dumbbells display different numbers of finite intervals of persistence with one much longer than the others. The dimpled dumbbell has some negligibly short finite persistence intervals and some longer ones. The scalar curvature death values of the longer finite intervals increase with each time step. All of the models have an infinite persistence interval of low birth scalar curvature; this is the only persistence interval present for the degenerate dumbbell. These results hold for different refinements of the triangulation; with higher spatial resolution, the curvature values increase, indicating neck pinching. The persistence diagrams appear somewhat self–similar across refinements. The dimension 1 persistence tables indicate increasing scalar curvature at each time step, but the scale is usually much lower than for the dimension 0 tables.

In the Betti number tables for all models but the degenerate dumbbell, multiple connected components and tunnels appear and disappear; we interpret this as indicative of noise. The presence of negligible persistence intervals appears associated with this. The longer intervals indicate increasing curvature, with multiple order–of–magnitude increases between birth and death values and between death values at each time step. The presence of these different classes of lengths of persistence intervals necessitates careful delineation of topological noise from legitimate topological signal.

We find from these outputs that persistent homology detects the formation and type of singularity across multiple spatial resolutions.

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