GUARANTEED QUALITY APPROXIMATIONS FOR MEDIAL AXIS OF IMPLICIT PLANAR CURVES

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Abstract. We apply subdivision methods to estimate the medial axis for a region bounded by an implicit curve $f(x) = 0$. If f has a continuous second derivative, then the output of the algorithm is homotopy equivalent to the the medial axis and is guaranteed to have Hausdorff distance at most ϵ to the actual medial axis for any $\epsilon > 0$.

1. Background

There are a variety of algorithms for approximating the medial axis. However, there are few exact algorithms, or ones with quality guarantees. All of these algorithms require restrictions on the shapes considered, for example, there are exact algorithms for finding the medial axis when the shape is a union of balls [1] or regions bounded by B-splines [4]. There are also a variety of general approximation algorithms, for example [2], that converge to the actual medial axis.

The prototypical example for approximating implicit curves and surfaces using subdivision methods is the marching cube algorithm of Lorensen and Cline [3]. In [5], the authors incorporated interval arithmetic into this algorithm, which guaranteed topological correctness of the output, a defect in the original algorithm. Recently, easily computable soft predicates have been used in subdivision algorithms to provide approximate solutions with precise correctness guarantees [6].

In this paper, we use soft predicates to perform calculations involving the medial axis. This allows us to construct a homotopy equivalent approximation of the medial axis for an implicit curve with any specified level of accuracy. In particular, these techniques can be applied to any shape in the plane whose boundary is given by a B-spline curve.

2. Algorithm

To perform calculations with the function $f(x)$, we assume that we have oracles that, for any square $S \subset \mathbb{R}^2$, can calculate intervals valued functions for $f(S)$, the gradient $\nabla f(S)$, and the curvature $\kappa(S)$. In particular, when f is a polynomial, these oracles can be developed with interval arithmetic.

The procedure uses a quadtree representation similar to the curve reconstruction algorithm in [5] and then builds the medial axis approximation. The nodes of the quadtree are subdivided until each square can be categorized as either intersecting the curve $C = \{x \mid f(x) = 0\}$, intersecting the medial axis or neither. The stopping criteria are summarized in the tale.

For a square R, in the interior of the region bounded by the curve, we must detect when R contains a point on the medial axis, i.e., has two closest points on C. For a point x in R, the closest points satisfy the equation $(f(y) - x) \cdot \nabla f(y) = 0$ with y on the curve. Using the oracles described above, we can find all nodes S of the quadtree where $0 \in (f(S) - R) \cdot \nabla f(S)$. This set is then pruned to only include nodes which might include global minima. These grid elements are combined to form collections of arcs, each being a candidate for containing the closest point to x.

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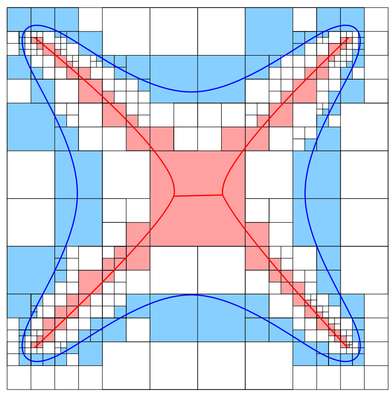
Conclusion	Stopping criteria
Intersects C	· Has two corners c_1, c_2 with $f(c_1)f(c_2) \leq 0$ · $0 \notin \nabla f(S)$ · $\lvert\kappa(D_S)\rvert < \frac{2}{\mathrm{Diag}(S)}$, where Diag($S$) is the length of the diagonal of S. · Every corner of another cell that has its closest point in C in the square has a unique closest point on C · Is not adjacent to a square intersecting the medial axis · S is adjacent to exactly two other such squares
Intersects medial axis	· Has two corners with closest points on different arcs of C, see discussion
Intersects neither	· $0 \notin f(S)$ · Every point in S has a unique closest point on C



For sufficiently small arcs, we can ensure that every point in R has a unique closest point on each of these arcs. This can be achieved using a curvature constraint.

After sufficiently many subdivisions have been performed, a graph approximating the medial axis can be built by forming a maximal forest in the dual graph of the nodes identified as intersecting the medial axis. If the boundary curve has multiple components, additional edges can be added to ensure topological correctness.

Theorem 1. *Given $f : \mathbb{R}^2 \to \mathbb{R}$, a square S and $\epsilon > 0$ such that*

(1) *f has a continuous second derivative*
(2) *The critical points of f do not lie on the curve C*
(3) *The position of S is generic*

then, the algorithm terminates and returns an approximation of the medial axis that is homotopy equivalent to and within Hausdorff distance ϵ of the exact medial axis interior to the shape.

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