Elucidating Peirce Quincuncial Projection

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Keywords: Peirce quinuncial projection, Elliptic Functions, Numerical Implementation , MAPLE.

Abstract

In this paper, we explain Peirce quincuncial projection, an almost-everywhere conformal projection suggested by the famous American philosopher Charles Sanders Peirce in 1879. This work expands the scant background provided in Pierce's paper. Firstly, we describe briefly the mathematics behind this map projection (in particular, some elements of Elliptic Functions). As a result of this step, we obtain a set of useful formulas. Then, we develop an algorithm to implement these formulas. Finally, we show some continents maps produced by our computer program implementing the algorithm.

Introduction

In general, a map is a representation of some space showing certain relations among the elements of such a space. In a more restricted sense, a map is a static 2D representation of a 3D space, usually the Earth surface. Maps exist from ancient times and there are many ways of making them. Cartography or map-making is the art of crafting representations of the Earth upon a flat surface. Actually, the word "*map* "comes from Latin mappa mundi, "*napkin or cloth of the world.* "

In 1879, the philosopher Charles Sanders Peirce proposed an angle-preserving map projection, i.e., a conformal map projection. It is very useful "for meteorological, magnetologcial and other purposes". This map also shows the connection of all parts on the Earth's surface.

Peirce's original paper [1]. is very scant of information. Perhaps because of this, some mathematicians tried, years later, to understand and explain his elegant idea. In particular, Pierpont [2]. detected an error in Peirce's formula and found a correct expression for the projetion. Other remarks were given shortly after by Frischauf [3].

In this work we briefly describe a theoretical method and its software implementation of the mathematical expressions found before.

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1 Background

The Jacobi elliptic functions are standard forms of elliptic functions. The basic functions are denoted $cn(z,k)$, $dn(z,k)$, and $sn(z,k)$, where k is known as the elliptic modulus and $z \in \mathbb{C}$. It arise from the inversion of the elliptic integral of the first kind denoted F.

The quincuncial projection is the composition of the stereographic projection with the cosine amplitude $(cn(z,k))$, in particular with $k = \frac{\sqrt{2}}{2}$.

Let $\theta \in (0, 2\pi)$, $l \in (-\pi/2, \pi/2)$, and $p = \pi/2 + l \in (0, \pi)$ be the longitude, latitude, and a parameter, distances respectively of a point P on the sphere of unit radius. So that the points of the North hemisphere, except the north pole, are projected on the equatorial plane when $cn(z, \frac{\sqrt{2}}{2}) = tan(\frac{p}{2})e^{i\theta}$.

2 Peirce's formula

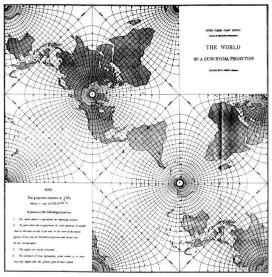
Perhaps Pierce was inspired in the conformal transformation of a circle onto a polygon of n sides, obtained by Schwarz in 1869, known as the Schwarz Christoffel mapping.

The formula given by Peirce for the quincuncial projection has the form $(\theta, l) \mapsto x$.

$$x = \frac{1}{2}F\left(arccos\sqrt{\frac{\sqrt{1-cos^2lcos^2\theta} - sinl}{1+\sqrt{1-cos^2lcos^2\theta}}}\right),$$

where x is "the value of one of the rectangular coordinates of the point in the new projection ".

Peirce's paper show the following illustrated.



3 Peirpont's formula

In your paper [2]. Peirpont said "*…There seems to be an error in this determination.* ", and using identities among Jacobi elliptic functions proof that the right formulas are as follows, we refered Solanilla [5] for details concerning the proof of identities.

Theorem 1. *Peirce quincuncial projection* $(\theta, p) \mapsto z = x + iy$ *is given by*

$$x = \frac{1}{2}F\left(arccos\frac{cos^2 l - 2\sqrt{sin^2 l + \frac{1}{4}cos^4 sin(2\theta)}}{2sinl + cos^2 lcos(2\theta)}\right),$$

$$y = \frac{1}{2}F\left(arccos\frac{2sinl + cos^2 lcos(2\theta)}{cos^2 l - 2\sqrt{sin^2 l + \frac{1}{4}cos^4 sin(2\theta)}}\right),$$

An equivalent expression can be obtained.

Theorem 2. *Peirce quincuncial projection* $(\theta, p) \mapsto z = x + iy$ *is given by*

$$x = \frac{1}{2}F\left(arccos\frac{2tan^2\frac{p}{2} - \sqrt{(1+tan^4\frac{p}{2})^2 - 4tan^4\frac{p}{2}cos^2 2\theta}}{1 + 2tan^2\frac{p}{2}cos(2\theta) - tan^4\frac{p}{2}}\right),$$

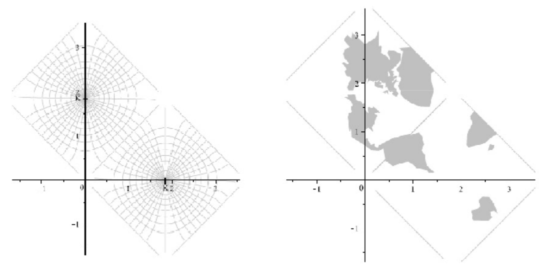
$$y = \frac{1}{2}F\left(arccos\frac{1 + 2tan^2\frac{p}{2}cos(2\theta) - tan^4\frac{p}{2}}{2tan^2\frac{p}{2} - \sqrt{(1+tan^4\frac{p}{2})^2 - 4tan^4\frac{p}{2}cos^2 2\theta}}\right),$$

4 Examples of the application of the algorithm in MAPLE

Let θ and l the geographical coordinates of a point on the sphere. The program chooses and computes the right formula according to the following table.

Latitude	Longitude	Octant	Shift
S	$0° \leq \theta < 90°\ E$	I	no
	$0° \leq \theta < 90°\ W$	II	no
	$90° \leq \theta < 180°\ E$	III	$\theta \leftarrow \theta - 90°$
	$90° \leq \theta < 180°\ W$	IV	$\theta \leftarrow \theta - 90°$
N	$0° \leq \theta < 90°\ E$	V	no
	$0° \leq \theta < 90°\ W$	VI	no
	$90° \leq \theta < 180°\ E$	VII	$\theta \leftarrow \theta - 90°$
	$90° \leq \theta < 180°\ W$	$VIII$	$\theta \leftarrow \theta - 90°$

The formula of Theorem 2 represents only points of the first octant, by symmetry is possible to represent all parts of the earth's surface



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