NP-hardness of the minimum point and edge 2-transmitter cover problem

Sarah Cannon* Thomas G. Fai† Justin Iwerks‡ Undine Leopold§ Christiane Schmidt¶

Abstract

We show it is NP-hard to compute a minimum cover of point 2-transmitters, point k-transmitters and edge 2-transmitters in a simple polygon; the point 2-transmitter result extends to orthogonal polygons.

Introduction.

The traditional art gallery problem (AGP) considers placing guards in an art gallery—modeled by a polygon—such that every point in the room can be seen by some guard. A similar question asks how to place wireless routers so that an entire room has a strong wireless signal. Observation shows that often not only the distance from a modem, but also the number of walls a signal has to pass, influences signal strength.

Aichholzer et al. [1] first formalized this problem by considering *k-modems*, also called *k-transmitters*, devices whose wireless signal can pass through at most k walls. Analogous to the AGP, two main questions can be considered: (1) Given a polygon P, can one efficiently find the minimum number of k-transmitters necessary to cover P? (2) Given a class of polygons of n vertices: What are lower and upper bounds on the number of guards that are needed to cover a polygon from this class? Here, we answer (1)—the complexity question.

Notations and Preliminaries.

A point $q \in P$ is *2-visible* from $p \in P$ if the straight-line connection $\overline{pq}$ intersects P in at most two connected components. For a point $p \in P$, we define the *2-visibility region* of p, $2\text{VR}(p)$, to be the set of all points in P that are 2-visible from p. For a set $S \subseteq P$, $2\text{VR}(S) := \cup_{p\in S} 2\text{VR}(p)$. A set $C \subseteq P$ is a *2-transmitter cover* if $2\text{VR}(C) = P$. Points used for a 2-transmitter cover are called *(point) 2-transmitters*. An *edge 2-transmitter e* can monitor all points of P that are 2-visible from some point of e.

NP-hardness results.

Minimum Point 2-Transmitter [k-transmitter] Cover (MP2TC) [MPkTC] Problem:

Given: A polygon P.

Task: Find the minimum cardinality 2-transmitter [k-transmitter] cover of P.

(a) (b)

Figure 1: (a) L in black and the resulting spike box in red. (b) The additional construction for 2-transmitters.

Theorem 1 *MP2TC is NP-hard for simple polygons.*

Proof. We reduce from the Minimum Line Cover Problem, shown to be NP-hard by Brodén et al. [3]:

Minimum Line Cover Problem (MLCP)

Given: A set L of non-parallel lines in the plane.

Task: Find the minimum set S of points such that there is at least one point in S on each line in L.

For a given set of lines L we construct a "spike box" P: an (axis-aligned) square Q that contains all intersection points plus two narrow spikes per line at the intersections with Q, see Fig. 1(a). If we consider computing the minimum 0-transmitter cover of P, at least one 0-transmitter must lie on each line. Hence, this problem is equivalent to a minimum line cover.

However, for the case of 2-transmitters we must slightly modify the spike-box construction. At each spike we add a small "crown": two additional spikes, resulting in a polygon $P(L)$. The spikes ensure that points in the central spike are visible only to points in Q along the original line from L, see Fig. 1(b). This yields: A minimum point cover of L is equivalent to a minimum 2-transmitter cover of $P(L)$. □

Observe that we can simply extend the above result to point-k-transmitters: we enlarge the "crowns" and add $k/2$ spikes to each side of the central spike that relates to the input line from L. This yields:

Corollary 2 *MPkTC is NP-hard for simple polygons.*

Theorem 3 *MP2TC is NP-hard for orthogonal, simple polygons.*

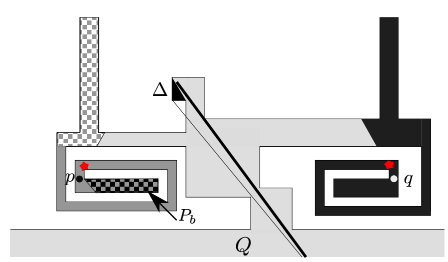

Figure 2: An orthogonal spike construction: $P(L)$ is shaded in light gray, a line of L is shown in bold black.

---

*College of Computing, Georgia Institute of Technology, Atlanta, USA. E-mail: sarah.cannon@gatech.edu. Supported by a Clare Boothe Luce Graduate Fellowship and NSF DGE-1148903.

†School of Engineering and Applied Sciences, Harvard University, MA, USA. E-mail: tfai@seas.harvard.edu

‡Mathematics Department, The Spence School, NY, USA. E-mail: jiwerks@gmail.com

§Mathematics Department, TU Chemnitz, Germany. E-mail: undine.leopold@mathematik.tu-chemnitz.de

¶The Rachel and Selim Benin School of Computer Science and Engineering, The Hebrew University of Jerusalem. E-mail: cschmidt@cs.huji.ac.il. Supported by the Israeli Centers of Research Excellence (I-CORE) program (Center No. 4/11).

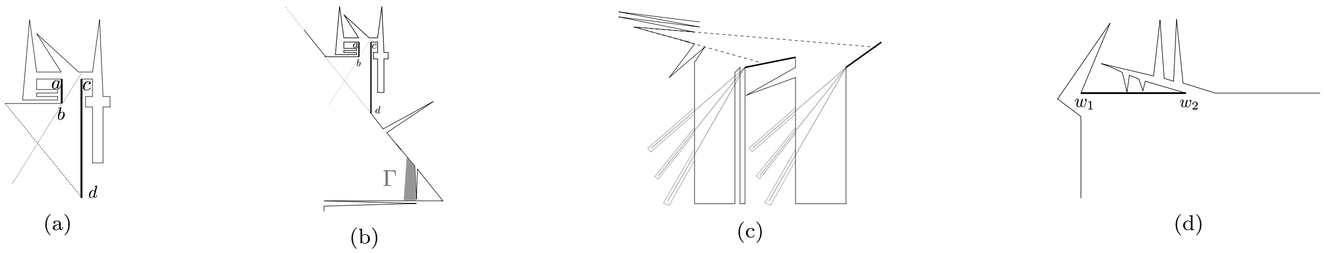

Figure 3: (a) Literal, (b) Clause and (c) Variable pattern, (d) Vertex W for edge 2-transmitters. In (b) a line of sight from the variable pattern spikes is indicated in gray.

Proof. Again we prove the statement by reduction from the MLCP. As Biedl et al. [2] showed in 2011, the MLCP remains NP-hard even if the given lines (parallelity allowed) have only one out of 4 slopes (horizontal, vertical, diagonal and off-diagonal in the octagonal grid), a problem we call MLCP4. The slope constraint ensures that our constructed polygon can be polynomially encoded.

Given the lines L of a MLCP4 instance, we rotate all lines by 22.5°. Again we construct an (axis-aligned) square Q that contains all intersection points. Due to the rotation, no line is orthogonal to an edge of Q. We construct a "spike box", but alter it slightly: For each spike we insert the construction shown in Fig. 2 (adapted to one of the four slopes), resulting in an orthogonal polygon $P_o(L)$. Let k be the number of lines in L, and ℓ the size of the MLCP4 solution.

We claim a minimum line cover of L of cardinality ℓ is equivalent to a minimum 2-transmitter cover of $P_o(L)$ of cardinality $\ell + 4k$.

In Fig. 2, p and q are only 2-visible from points in the regions shaded dark gray and black, respectively, and not from Q. In addition, the black triangle, Δ, is 2-visible from no other spike gadget, and in Q only from points along the thickened line representing the line from L. If we want to place a 2-transmitter g to simultaneously cover p and Δ, g must be located in the polygon area with white squares. But then, no point in the area with black squares, P_b, is visible to g. No point in Q covers P_b, requiring an additional 2-transmitter within the spike gadget. An analogous argument is used for q. Consequently, the minimum number of 2-transmitters that cover the spike gadget—except for Δ, and parts of the lower corridors of the two spirals which are 2-visible from any point within Q—is 2: located at the 2 (red) stars. This results in $4k$ 2-transmitters (2 per spike).

The remaining Δ's in all spike gadgets can be covered with ℓ 2-transmitters iff the lines of L can be covered by ℓ points, which establishes the claim. □

Minimum Edge 2-Transmitter Cover (ME2TC):
Given: A polygon P.
Task: Find the minimum cardinality edge 2-transmitter cover of P.

Theorem 4 *ME2TC is NP-hard for simple polygons.*

Proof. We adapt the proof for the minimum edge guard problem of Lee and Lin[4], i.e., their reduction from 3SAT. As with their adaptation of the point guard AGP, we need to modify the literal pattern, the variable pattern, the vertex W and in our case also the clause pattern, see Fig. 3. For the variable pattern we make the additional assumption that each literal appears in at least 3 clauses (otherwise we can add spikes to the rectangle wells of the variable pattern).

In Fig. 3(a) only 2-transmitters $\overline{ab}$ and $\overline{cd}$ can cover the entire literal pattern; they correspond to truth settings false and true, resp.. In Fig. 3(b) the spikes around region Γ ensure (i) edges $\overline{ab}$ corresponding to a truth setting of false cannot cover Γ, and (ii) no edge outside the clause gadget can cover all of Γ—we add the same construction on the other side of the clause triangle. In Fig. 3(c) only the two bold 2-transmitters cover the lowest of the 3 consecutive triangles to the left and also one of the wells and its spikes in the variable pattern. We add two additional wells, visible from w_1, between variable patterns, preventing edges inside a variable pattern from monitoring another. We replace vertex W by edge $\overline{w_1w_2}$, see Fig. 3(d). Point w_1 serves the same purpose as W in the original proof; only $\overline{w_1w_2}$ covers both the entire attached gadget (necessary to prevent choosing edges adjacent to $\overline{w_1w_2}$, which may cover more than desired) and all wells in and between variable gadgets. □

Conclusion. In addition to (1), answered here, in future work we address (2) and present new results for both point 2-transmitters and edge 2-transmitters.

Acknowledgments. We would like to thank Joseph O'Rourke for sharing the 2-transmitter problem with us during the Mathematics Research Communities workshop on Discrete and Computational Geometry in 2012 and the AMS for their financial support of this program.

References

[1] O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, J. Urrutia, and B. Vogtenhuber. Modem illumination of monotone polygons. In *Proc. 25th Europ. Workshop Comp. Geom.*, pages 167–170, 2009.

[2] T. C. Biedl, M. T. Irfan, J. Iwerks, J. Kim, and J. S. B. Mitchell. Guarding polyominoes. In *Proc. 27th ACM Sympos. on Comp. Geom.*, pages 387–396, 2011.

[3] B. Brodén, M. Hammar, and B. J. Nilsson. Guarding lines and 2-link polygons is apx-hard. In *Proc. 13th Canad. Conf. Comput. Geom.*, pages 45–48, 2001.

[4] D.-T. Lee and A. K. Lin. Computational complexity of art gallery problems. *IEEE Trans. Inf. Theor.*, 32(2):276–282, 1986.