Extensions of Golomb's Tromino Theorem

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Introduction

A polyomino is a finite edge–connected collection of equal-sized squares in the plane [1]. Often the stipulation is added that the union of the collection of squares have no holes. A tromino is a polyomino consisting of just three squares. One of the simplest and most beautiful theorems about polyominos is Solomon Golomb's Tromino Theorem, which states that if you start with a chess board of size 2^N x 2^N and remove one of the squares, then the remaining board can always be covered by trominos of the type shown in Figure 1, which we call the "basic tromino." In this paper we

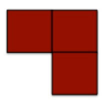


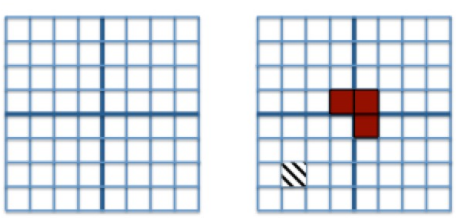
Figure 1. The basic tromino.

show how Golomb's Theorem can be generalized to three and higher dimensions and then give versions of Golomb's Theorem that hold on boards of size 3^N x 3^N and 4^N x 4^N.

1 Proof of Golomb's Theorem

Theorem 1 *For any integer $N \geq 0$, if we remove a single square from a chess board of size 2^N x 2^N, the remaining board can entirely be tiled by basic trominos.*

Proof. The proof is true for the case $N = 0$ since after removing the one square there is nothing to cover. Now consider a 2^N x 2^N board for $N \geq 1$, and divide this board into 2 x 2 array of 2^{N-1} x 2^{N-1} boards as shown in Figure 2A. Any



A B

Figure 2. A. A 2^N x 2^N board divided into 2 x 2 array of 2^{N-1} x 2^{N-1} boards. B. After a square has been removed from the bottom-left 2^{N-1} x 2^{N-1} board a tromino is placed such that a square is taken from each of the other three 2^{N-1} x 2^{N-1} boards. Each of the smaller boards may then be covered by basic trominos by induction.

square removed from the 2^N x 2^N board must fall in one of the four smaller 2^{N-1} x 2^{N-1} boards. Without loss of generality, we suppose it is the board on the bottom left. Now place a basic tromino as shown in Figure 2B and complete the proof by tiling each of the 2^{N-1} x 2^{N-1} boards, now with one tile missing, by induction. □

2 3D and Higher Versions of Golomb's Theorem

A $3D$ polyomino is a finite, face–connected collection of equal-sized cubes in 3-space, with the natural extension to N-dimensional polyominoes for any N.

The extension of Golomb's Theorem to three and higher dimensions is not difficult. In dimension three if we remove one unit-edge length cube from a 2^N x 2^N x 2^N three dimensional chessboard the remaining board can be covered by 7–cell $3D$–polyominoes. We chop a 2 x 2 x 2 cube into eight unit sub-cubes, and throw one away. Now if we remove one cubic cell from the 2^N x 2^N x 2^N board, and think of this board as a collection of eight 2^{N-1} x 2^{N-1} x 2^{N-1} smaller boards, one per octant, then we can place our fundamental 7–3D-poyomino such that it has one of its component cubes in each of the smaller boards in which we did not remove a cube. Just like in the 2D case, the smaller boards are then all coverable by induction.

In dimension M the argument is the same. Formally:

Theorem 2 *Consider an M-dimensional chessboard of edge length 2^N. If we remove a single unit edge-length M-dimensional hypercube from this hyper board then the remaining board can be covered by $M-$dimensional polyominoes consisting of an M-dimensional hypercube of of edge length 2 with one 1x1 sub-hypercube removed.*

3 Covering 3^N x 3^N Boards

Before seeing Golomb's clever inductive covering argument for the 2^N x 2^N board, it was perhaps not immediately obvious that 3 actually divides $2^{2N} - 1$. However,

$$2^{2N} - 1 = 4^N - 1 = (4-1)(4^{N-1} + 4^{N-2} + ... + 1).$$

When considering the analogous covering problem for a 3^N x 3^N board we have

$$3^{2N} - 1 = 9^N - 1 = (9-1)(9^{N-1} + 9^{N-2} + ... + 1).$$

Thus we were tempted to look for tilings of a 3^N x 3^N board with one square removed by 8-polyominoes and 4-polyominoes (in other words, by octominoes and tetrominoes). Indeed, it is not difficult to convince oneself that hexominoes do not work so well. However, we have:

Theorem 3 *One can completely tile a 3^N x 3^N chess board, with one square removed, with tetrominoes of the three types depicted in Figure 3.*

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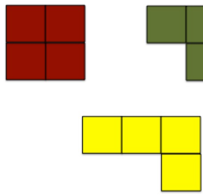

Figure 3. The three basic tetrominoes used for covering a 3^N x 3^N board with one square removed.

Proof. (1) The case $N = 0$ is obvious; after removing a square there is nothing to cover.
(2) Verify by hand for N = 1 (up to symmetry there are three tiles that can be removed)
(3) Consider the basic tetrominoes but where each of the 4 squares actually consists of 3^M x 3^M smaller squares. Prove by induction that these shapes can each be tiled by the three basic tetrominoes. See Figure 4. (4) Now consider a 3^N

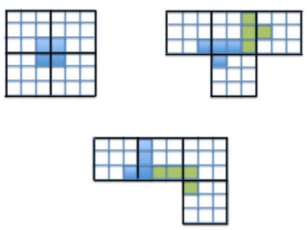

Figure 4. Induction step showing that each of the basic tetrominoes consisting of 3^M x 3^M squares can be covered by basic tetrominoes.

x 3^N chess board which we draw as a 3x3 set of 3^{N-1} x 3^{N-1} chess boards. Assume, by induction, that the conjecture holds for 3^{N-1} x 3^{N-1} chess boards. If we remove one of the squares from this board, the square must come from one of the 9 3^{N-1} x 3^{N-1} chess boards. By induction we can cover the rest of this 3^{N-1} x 3^{N-1} chess board. But the remaining 8 3^{N-1} x 3^{N-1} chess boards can be covered by the basic shapes considered in step (3) [this is just the case $n = 1$ again]. By the argument is step (3) the basic shapes can be covered, so the entire rest of the chess board can be covered. □

4 Covering 4^N x 4^N Boards

For 4^N x 4^N boards we have a similar result, though something a bit stronger can be said:

Theorem 4 *One can completely tile a 4^N x 4^N chess board, with one square removed, with an equal number of the pentominoes of the three types depicted in Figure 5.*

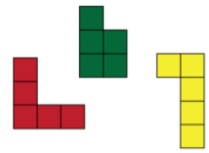

Figure 5. The three basic pentominoes used for covering a 4^N x 4^N board with one square removed.

Proof. Up to symmetry there are just three ways of removing a square from a 4 x 4 chess board, and as Figure 6 shows, the theorem holds in these cases. For the general 4^N x 4^N case, the proof follows the same general steps as in the proof of the 3^N x 3^N problem. However, in step (3) when we show that each of the basic pentominoes can themselves be

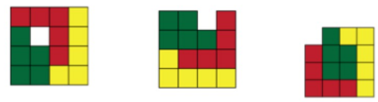

Figure 6. Covering the three different 4 x 4 boards with one each of the basic pentominoes.

covered, we do not actually show that they can be covered using an equal number of each of the basic pentominoes, but rather that collectively the pentominoes can be so covered. Figure 7 illustrates this inductive step. □

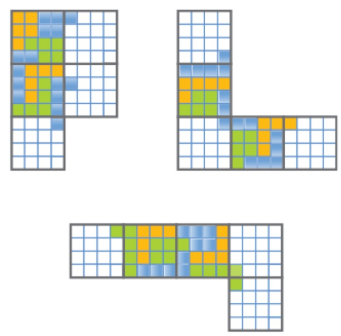

Figure 7. Covering an equal number of basic pentominoes, with 4^M x 4^M squares per block, collectively with an equal number of basic pentominoes.

Concluding Remarks

We have just scratched the surface regarding the possible extensions of Golomb's Theorem. We intend to push our inquiry on to 5^N x 5^N boards with the hope of obtaining a general result for boards of size K^N x K^N, for arbitrary K. We note that, up to symmetry, there was just one way of removing a square from a 2 x 2 board and three ways to remove squares from both a 3 x 3 and 4 x 4 board. In each case this number turned out to be the magic number of basic polyominoes needed to cover. Does this correspondence continue? Can anything interesting be said about covering a 3^N x 3^N with 8–polyominoes or the 4^N x 4^N board with 15–polyominoes? Finally, are there extensions of our 3^N x 3^N and 4^N x 4^N results to three and higher dimensions?

References

[1] Solomon W. Golomb. *Polyominoes: Puzzles, Patterns, Problems, and Packings*. Princeton University Press, 2nd edition, 1996.